ON SOME FUNDAMENTAL THEOREMS OF MENSURATION IN ONE, TWO, AND THREE DIMENSIONS.*

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This article has for its subject the following three analogies:

$$L = \frac{1}{1!}\begin{vmatrix} x_2 - x_1 \end{vmatrix}, \tag{1}$$

$$A = \frac{1}{2!}\begin{vmatrix} x_4 - x_2, & x_3 - x_1 \\ y_4 - y_2, & y_3 - y_1 \end{vmatrix}, \tag{2}$$

$$V = \frac{1}{3!}\begin{vmatrix} x_6 - x_3, & x_5 - x_2, & x_4 - x_1 \\ y_6 - y_3, & y_5 - y_2, & y_4 - y_1 \\ z_6 - z_3, & z_5 - z_2, & z_4 - z_1 \end{vmatrix} \tag{3}$$

The first of these formulæ gives the distance of two points whose distances from the origin x_1 and x_2 are given.

The second gives the area of any quadrilateral in terms of the co-ordinates of its vertices.

Similarly, the third gives the volume of any octahedron, or any solid within six points in space, in terms of orthogonal co-ordinates.

It will be noticed that in place of the usual fundamental forms, distance, triangle, and tetrahedron, we have here distance, quadrilateral, and octahedron, and it will be easily seen that above general formulæ can be used for triangle and tetrahedron by causing one or two points respectively to coincide. This does not, however, essentially simplify the formulæ. We have thus for the area of triangle 1. 2. 3

$$A = \frac{1}{2!}\begin{vmatrix} x_3 - x_2, & x_3 - x_1 \\ y_3 - y_2, & y_3 - y_1 \end{vmatrix} = \frac{1}{2!}\begin{vmatrix} x_3 - x_2, & x_2 - x_1 \\ y_3 - y_2, & y_2 - y_1 \end{vmatrix} = \frac{1}{2!}\begin{vmatrix} x_2 - x_1, & x_1 - x_3 \\ y_2 - y_1, & y_1 - y_3 \end{vmatrix}, \tag{4}$$

and a variety of other forms obtained by transformation.

We have also for the volume of the tetrahedron 1. 2. 3. 4

$$V = \frac{1}{3!}\begin{vmatrix} x_4 - x_3, & x_4 - x_2, & x_4 - x_1 \\ y_4 - y_3, & y_4 - y_2, & y_4 - y_1 \\ z_4 - z_3, & z_4 - z_2, & z_4 - z_1 \end{vmatrix}$$

$$= \frac{1}{3!}\begin{vmatrix} x_4 - x_3, & x_3 - x_2, & x_2 - x_1 \\ y_4 - y_3, & y_3 - y_2, & y_2 - y_1 \\ z_4 - z_3, & z_3 - z_2, & z_2 - z_1 \end{vmatrix} \tag{5}$$

$= \text{etc.}$

* Read before the Mathematical Section of the Philosophical Society of Washington, D. C.

On the other hand, it will appear from the demonstration of these formulæ that neither the area of any other polygon nor the volume of any other polyhedron can be expressed by a single determinant. The area of any polygon of n sides is the sum of n trapezoids formed by each side with its projection on one of the axes, taking it positive if in following the contour we go farther from the other axis, and negative if we approach it. The area of the trapezoid on the side 1. 2 is $= \frac{1}{2}(x_2 - x_1)(y_2 + y_1)$, and that of the other trapezoids is obtained with the proper sign by simply circulating subscripts in the order: 1, 2, 3, . . . n, 1. Thus we have

$$
\begin{aligned}
A = &\ \tfrac{1}{2}(x_2 - x_1)(y_2 + y_1) + \tfrac{1}{2}(x_3 - x_2)(y_3 + y_2) + \tfrac{1}{2}(x_4 - x_3)(y_4 + y_3) \\
&+ \tfrac{1}{2}(x_5 - x_4)(y_5 + y_4) + \ \ldots \ + \tfrac{1}{2}(x_1 - x_n)(y_1 + y_n) \\
= &\ \tfrac{1}{2}\begin{vmatrix} x_2, & x_1 \\ y_2, & y_1 \end{vmatrix} + \tfrac{1}{2}\begin{vmatrix} x_3, & x_2 \\ y_3, & y_2 \end{vmatrix} + \tfrac{1}{2}\begin{vmatrix} x_4, & x_3 \\ y_4, & y_3 \end{vmatrix} + \tfrac{1}{2}\begin{vmatrix} x_5, & x_4 \\ y_5, & y_4 \end{vmatrix} + \ \ldots \ + \tfrac{1}{2}\begin{vmatrix} x_1, & x_n \\ y_1, & y_n \end{vmatrix} \\
= &\ \tfrac{1}{2}\begin{vmatrix} x_3 - x_1, & x_2 \\ y_3 - y_1, & y_2 \end{vmatrix} + \tfrac{1}{2}\begin{vmatrix} x_5 - x_3, & x_4 \\ y_5 - y_3, & y_4 \end{vmatrix} + \ \ldots\ .
\end{aligned}
$$

If n is even, the area is thus expressed by $\frac{1}{2}n$ determinants, which are not combinable, in general, because they have no common column. In the case of the quadrilateral, 5 coincides with 1, and we have

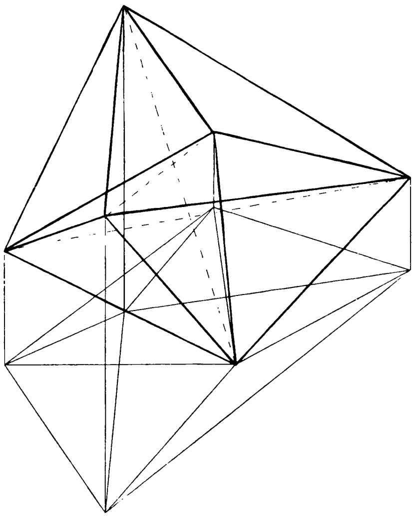
$$
\begin{aligned}
A &= \tfrac{1}{2}\begin{vmatrix} x_3 - x_1, & x_2 \\ y_3 - y_1, & y_2 \end{vmatrix} + \tfrac{1}{2}\begin{vmatrix} x_1 - x_3, & x_4 \\ y_1 - y_3, & y_4 \end{vmatrix} \\
&= \tfrac{1}{2}\begin{vmatrix} x_4 - x_2, & x_3 - x_1 \\ y_4 - y_2, & y_3 - y_1 \end{vmatrix}.
\end{aligned}
$$

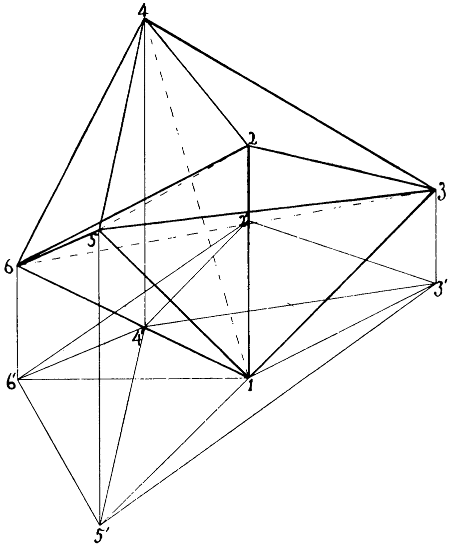
In using this formula care should be taken that the elements of this determinant are the projections of *internal* diagonals or in case of a re-entrant qualrilateral at least of one internal diagonal, for if we take the points in a different order we obtain the area of a crossed quadrilateral. If these rules are attended to we obtain the area of a quadrilateral in the ordinary sense except with regard to sign. If we wish it positive we must figure the vertices so that the positive diagonal product $(x_4 - x_2)(y_3 - y_1)$ is positive and as large as possible.

Formula (2) is not essentially new, except, perhaps, being expressed by a determinant, for it may be easily recognized in a rule for computing area as used in land surveying.*

Formula (3) may be demonstrated in an analogous manner. For any polyhedron may be expressed by the algebraic sum of as many trapezoidal prisms

* See, for instance, Davies and Peck's "Dictionary of Mathematics," Article Survey, p. 550.




4
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as it has faces, taking those negative which are wholly external. It is of course immaterial on which co-ordinate plane the faces are projected, but we shall use the horizontal plane.

Suppose we wish to have the volume of a trapezoidal prism formed by projecting the face (m, n, o) on the horizontal plane. It is well known to be = projection $\times$ mean height, or, using formula (4), we have

$$P_{m.n.o} = \frac{1}{2!}\begin{vmatrix} x_m - x_n, & x_n - x_o \\ y_m - y_n, & y_n - y_o \end{vmatrix} \times \frac{z_m + z_n + z_o}{3}$$

$$= \frac{1}{2!}\begin{vmatrix} x_n - x_o, & x_o - x_m \\ y_n - y_o, & y_o - y_m \end{vmatrix} \times \frac{z_n + z_o + z_m}{3}$$

$$= \frac{1}{2!}\begin{vmatrix} x_o - x_m, & x_m - x_n \\ y_o - y_m, & y_m - y_n \end{vmatrix} \times \frac{z_o + z_m + z_n}{3},$$

or we may write

$$P_{m.n.o} = \frac{1}{3!}\begin{vmatrix} x_m - x_n, & x_n - x_o, & x_o - x_m \\ y_m - y_n, & y_n - y_o, & y_o - y_m \\ z_o, & z_m, & z_n \end{vmatrix}, \tag{6}$$

where the third row may also be written in any order whatever, because their minors are equivalent. If we reverse the order of the points, we have

$$P_{o.n.m} = \frac{1}{3!}\begin{vmatrix} x_o - x_n, & x_n - x_m, & x_m - x_o \\ y_o - y_n, & y_n - y_m, & y_m - y_o \\ z_m, & z_o, & z_n \end{vmatrix} = -P_{m.n.o}. \tag{7}$$

Now it is clear that if a trapezoidal prism is external to a polyhedron, then the order of the points on the latter is the reverse to that on the former. If, therefore, in using (6) for the trapezoidal prisms on each face of a polyhedron we take the points always in the order as they appear on the latter, we obtain their volumes with the proper sign, and their sum total will be the volume of the polyhedron.

To aid the perception I have constructed a stereoscopic diagram of an octahedron, exhibiting also the eight trapezoidal prisms $P_{1.2.3}$, $P_{1.3.5}$, $P_{1.5.6}$, $P_{1.6.2}$, $P_{4.3.2}$, $P_{4.5.3}$, $P_{4.6.5}$, $P_{4.2.6}$, and we have the volume,

$$V = \frac{1}{3!}\begin{vmatrix} x_1 - x_2, & x_2 - x_3, & x_3 - x_1 \\ y_1 - y_2, & y_2 - y_3, & y_3 - y_1 \\ z_3, & z_1, & z_2 \end{vmatrix} + \frac{1}{3!}\begin{vmatrix} x_1 - x_3, & x_3 - x_5, & x_5 - x_1 \\ y_1 - y_3, & y_3 - y_5, & y_5 - y_1 \\ z_5, & z_1, & z_3 \end{vmatrix}$$

$$+\frac{1}{3!}\begin{vmatrix} x_1-x_5, & x_5-x_6, & x_6-x_1 \\ y_1-y_5, & y_5-y_6, & y_6-y_1 \\ z_6, & z_1, & z_5 \end{vmatrix}+\frac{1}{3!}\begin{vmatrix} x_1-x_6, & x_6-x_2, & x_2-x_1 \\ y_1-y_6, & y_6-y_2, & y_2-y_1 \\ z_2, & z_1, & z_6 \end{vmatrix}$$

$$+\frac{1}{3!}\begin{vmatrix} x_4-x_3, & x_3-x_2, & x_2-x_4 \\ y_4-y_3, & y_3-y_2, & y_2-y_4 \\ z_2, & z_4, & z_3 \end{vmatrix}+\frac{1}{3!}\begin{vmatrix} x_4-x_5, & x_5-x_3, & x_3-x_4 \\ y_4-y_5, & y_5-y_3, & y_3-y_4 \\ z_3, & z_4, & z_5 \end{vmatrix}$$

$$+\frac{1}{3!}\begin{vmatrix} x_4-x_6, & x_6-x_5, & x_5-x_4 \\ y_4-y_6, & y_6-y_5, & y_5-y_4 \\ z_5, & z_4, & z_6 \end{vmatrix}+\frac{1}{3!}\begin{vmatrix} x_4-x_2, & x_2-x_6, & x_6-x_4 \\ y_4-y_2, & y_2-y_6, & y_6-y_4 \\ z_6, & z_4, & z_2 \end{vmatrix}$$

Arranging this in terms of z_1, z_2, z_3, z_4, z_5, and z_6, we have

$$\begin{aligned}
V= \quad & \frac{z_1}{3!}\left\{\begin{vmatrix} x_3-x_1, & x_1-x_2 \\ y_3-y_1, & y_1-y_2 \end{vmatrix}+\begin{vmatrix} x_5-x_1, & x_1-x_3 \\ y_5-y_1, & y_1-y_3 \end{vmatrix}\right\} \\
+ & \frac{z_1}{3!}\left\{\begin{vmatrix} x_6-x_1, & x_1-x_5 \\ y_6-y_1, & y_1-y_5 \end{vmatrix}+\begin{vmatrix} x_2-x_1, & x_1-x_6 \\ y_2-y_1, & y_1-y_6 \end{vmatrix}\right\} \\
+ & \frac{z_2}{3!}\left\{\begin{vmatrix} x_1-x_2, & x_2-x_3 \\ y_1-y_2, & y_2-y_3 \end{vmatrix}+\begin{vmatrix} x_6-x_2, & x_2-x_1 \\ y_6-y_2, & y_2-y_1 \end{vmatrix}\right\} \\
+ & \frac{z_2}{3!}\left\{\begin{vmatrix} x_3-x_2, & x_2-x_4 \\ y_3-y_2, & y_2-y_4 \end{vmatrix}+\begin{vmatrix} x_4-x_2, & x_2-x_6 \\ y_4-y_2, & y_2-y_6 \end{vmatrix}\right\} \\
+ & \frac{z_3}{3!}\left\{\begin{vmatrix} x_2-x_3, & x_3-x_1 \\ y_2-y_3, & y_3-y_1 \end{vmatrix}+\begin{vmatrix} x_1-x_3, & x_3-x_5 \\ y_1-y_3, & y_3-y_5 \end{vmatrix}\right\} \\
+ & \frac{z_3}{3!}\left\{\begin{vmatrix} x_4-x_3, & x_3-x_2 \\ y_4-y_3, & y_3-y_2 \end{vmatrix}+\begin{vmatrix} x_5-x_3, & x_3-x_4 \\ y_5-y_3, & y_3-y_4 \end{vmatrix}\right\} \\
+ & \frac{z_4}{3!}\left\{\begin{vmatrix} x_2-x_4, & x_4-x_3 \\ y_2-y_4, & y_4-y_3 \end{vmatrix}+\begin{vmatrix} x_3-x_4, & x_4-x_5 \\ y_3-y_4, & y_4-y_5 \end{vmatrix}\right\} \\
+ & \frac{z_4}{3!}\left\{\begin{vmatrix} x_5-x_4, & x_4-x_6 \\ y_5-y_4, & y_4-y_6 \end{vmatrix}+\begin{vmatrix} x_6-x_4, & x_4-x_2 \\ y_6-y_4, & y_4-y_2 \end{vmatrix}\right\} \\
+ & \frac{z_5}{3!}\left\{\begin{vmatrix} x_3-x_5, & x_5-x_1 \\ y_3-y_5, & y_5-y_1 \end{vmatrix}+\begin{vmatrix} x_1-x_5, & x_5-x_6 \\ y_1-y_5, & y_5-y_6 \end{vmatrix}\right\} \\
+ & \frac{z_5}{3!}\left\{\begin{vmatrix} x_4-x_5, & x_5-x_3 \\ y_4-y_5, & y_5-y_3 \end{vmatrix}+\begin{vmatrix} x_6-x_5, & x_5-x_4 \\ y_6-y_5, & y_5-y_4 \end{vmatrix}\right\} \\
+ & \frac{z_6}{3!}\left\{\begin{vmatrix} x_5-x_6, & x_6-x_1 \\ y_5-y_6, & y_6-y_1 \end{vmatrix}+\begin{vmatrix} x_1-x_6, & x_6-x_2 \\ y_1-y_6, & y_6-y_2 \end{vmatrix}\right\}
\end{aligned}$$

$$+\frac{z_6}{3!}\left\{\begin{vmatrix} x_4-x_6, & x_6-x_5 \\ y_4-y_6, & y_6-y_5 \end{vmatrix}+\begin{vmatrix} x_2-x_6, & x_6-x_4 \\ y_2-y_6, & y_6-y_4 \end{vmatrix}\right\}$$

$$=\frac{z_1}{3!}\left\{\begin{vmatrix} x_3-x_1, & x_5-x_2 \\ y_3-y_1, & y_5-y_2 \end{vmatrix}+\begin{vmatrix} x_6-x_1, & x_2-x_5 \\ y_6-y_1, & y_2-y_5 \end{vmatrix}\right\}$$

$$+\frac{z_2}{3!}\left\{\begin{vmatrix} x_1-x_2, & x_6-x_3 \\ y_1-y_2, & y_6-y_3 \end{vmatrix}+\begin{vmatrix} x_4-x_2, & x_3-x_6 \\ y_4-y_2, & y_3-y_6 \end{vmatrix}\right\}$$

$$+\frac{z_3}{3!}\left\{\begin{vmatrix} x_2-x_3, & x_4-x_1 \\ y_2-y_3, & y_4-y_1 \end{vmatrix}+\begin{vmatrix} x_5-x_3, & x_1-x_4 \\ y_5-y_3, & y_1-y_4 \end{vmatrix}\right\}$$

$$+\frac{z_4}{3!}\left\{\begin{vmatrix} x_3-x_4, & x_2-x_5 \\ y_3-y_4, & y_2-y_5 \end{vmatrix}+\begin{vmatrix} x_6-x_4, & x_5-x_2 \\ y_6-y_4, & y_5-y_2 \end{vmatrix}\right\}$$

$$+\frac{z_5}{3!}\left\{\begin{vmatrix} x_1-x_5, & x_3-x_6 \\ y_1-y_5, & y_3-y_6 \end{vmatrix}+\begin{vmatrix} x_4-x_5, & x_6-x_3 \\ y_4-y_5, & y_6-y_3 \end{vmatrix}\right\}$$

$$+\frac{z_6}{3!}\left\{\begin{vmatrix} x_2-x_6, & x_1-x_4 \\ y_2-y_6, & y_1-y_4 \end{vmatrix}+\begin{vmatrix} x_5-x_6, & x_4-x_1 \\ y_5-y_6, & y_4-y_1 \end{vmatrix}\right\}$$

$$=\frac{z_4-z_1}{3!}\begin{vmatrix} x_6-x_3, & x_5-x_2 \\ y_6-y_3, & y_5-y_1 \end{vmatrix}+\frac{z_5-z_2}{3!}\begin{vmatrix} x_4-x_1, & x_6-x_3 \\ y_4-y_1, & y_6-y_3 \end{vmatrix}+\frac{z_6-z_3}{3!}\begin{vmatrix} x_5-x_2, & x_4-x_1 \\ y_5-y_2, & y_4-y_1 \end{vmatrix}$$

$$=\frac{1}{3!}\begin{vmatrix} x_6-x_3, & x_5-x_2, & x_4-x_1 \\ y_6-y_3, & y_5-y_2, & y_4-y_1 \\ z_6-z_3, & z_5-z_2, & z_4-z_1 \end{vmatrix}. \quad \text{Q. E. D.}$$

If the co-ordinates are measured along oblique axes, so that the y-axis inclines to the x-axis by the angle y_x, and the z-axis to the xy-plane by the angle z_{xy}, then, if we replace in all preceding formulæ

$$y \text{ by } y \sin y_x,$$
$$z \text{ by } z \sin z_{xy};$$

we have the formulæ for oblique co-ordinates

$$L=\frac{1}{1!}\begin{vmatrix} x_2-x_1 \end{vmatrix}, \tag{1'}$$

$$A=\frac{1}{2!}\begin{vmatrix} x_4-x_2, & x_3-x_1 \\ y_4-y_2, & y_3-y_1 \end{vmatrix}\sin y_x, \tag{2'}$$

$$V=\frac{1}{3!}\begin{vmatrix} x_6-x_3, & x_5-x_2, & x_4-x_1 \\ y_6-y_3, & y_5-y_2, & y_4-y_1 \\ z_6-z_3, & z_5-z_2, & z_4-z_1 \end{vmatrix}\sin y_x \sin z_{xy}; \tag{3'}$$

and if the x-axis coincides with diagonals 2. 4 of the quadrilateral, and 3. 6 of the octahedron, the y-axis with diagonals 1. 3 and 5. 2, and the z-axis with diagonal 1. 4, then the elements in the positive diagonal of the determinants are the respective diagonals, while all side elements vanish, and we have

$$L = \frac{1}{1!}(1.\ 2), \tag{1''}$$

$$A = \frac{1}{2!}(2.\ 4)\,(1.\ 3)\sin y_x, \tag{2''}$$

$$V = \frac{1}{3!}(3.\ 6)\,(2.\ 5)\,(1.\ 4)\sin y_x \sin z_{xy}. \tag{3''}$$

SOLUTIONS OF EXERCISES.

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SHOW that the locus of the centres of equilateral hyperbolas circumscribed to a given triangle is the nine-points circle of the triangle.

SOLUTION.

The given triangle is the triangle of reference. The circumconic

$$u\beta\gamma + v\gamma\alpha + w\alpha\beta = 0$$

will be an equilateral hyperbola if

$$u \cos A + v \cos B + w \cos C = 0;$$

$$v\gamma + w\beta \ :\ w\alpha + u\gamma \ :\ u\beta + v\alpha = a \ :\ b \ :\ c$$

are the equations to the centre; and the resultant of this system is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = a\alpha^2 \cos A + b\beta^2 \cos B + c\gamma^2 \cos C,$$

which represents the nine-points circle. In determinant notation this resultant can be written

$$\begin{vmatrix} 0 & \alpha & \beta & c \\ \alpha & 0 & \gamma & b \\ \beta & \gamma & 0 & a \\ \cos C & \cos B & \cos A & 0 \end{vmatrix} = 0,$$

a form which is probably new. [*R. H. Graves.*]